Grid Diagrams of Knots

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Abstract

In this article, we will describe grid diagrams of knots and their applications to more general knot theory. In particular, we describe grid moves and Cromwell's Theorem, which relates grid diagrams by knot class.

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1 Introduction

Grid diagrams are a presentations of link diagrams, which are a combinatorial way to represent links.[1] More recently, they have found much use as a combinatorial interpretation of knot Floer homology, which categorifies the Alexander polynomial. The main reference for this section is [OSS15].

Definition 1.1 (Grid Diagrams). A (planar) **grid diagram** $\mathbf{G}$ is an $n \times n$ planar grid that satisfies the following properties:

i Exactly n squares are marked with a $\circ$ and another n are marked with a $\bullet$.

ii Each row has a single square marked with a $\circ$ and another marked with a $\bullet$.

iii Each column has a single square marked with a $\circ$ and another marked with a $\bullet$.

iv No square is marked with both.

The number n above is the **grid number** of $\mathbf{G}$. To obtain a specific oriented link L from $\mathbf{G}$, we use the following procedure:

Algorithm 1.2 (Link Recovery). Start with the diagram $\mathbf{G}$.

i Connect the $\circ$ marked squares to the $\bullet$ marked squares in each column with *oriented* segments.

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[1] In particular, this facilitates the input of links into computer programs for algorithmic processing and even machine learning, see [Guk+20] as an example.

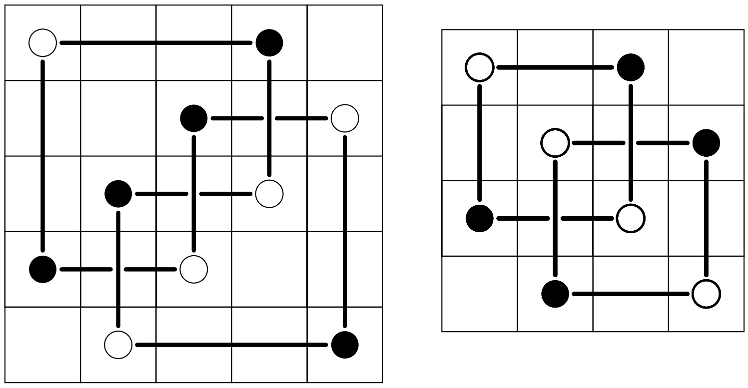

Figure 1: A grid diagram for the right handed trefoil $T(2,3)$ and the Hopf Link $\widehat{\sigma_1^2}$.

ii Perform the same procedure for rows. Vertical segments should always cross over horizontal ones.

iii The resulting diagram is the oriented link diagram L represented by G.

Refer to fig. 1 for an example of grid diagrams for specified links. This algorithm naturally leads to the question of whether every oriented link can be represented by a grid diagram, which is true.

Theorem 1.3. Every oriented link (embedded into S^3) can be represented by a grid diagram.

Proof. Given an oriented diagram of a link L, we can approximate it with a piecewise linear embedding[2], and perturb it such that there are only horizontal and vertical segments. Since by conventions all over-crossings in a grid diagram are by vertical segments, we must modify any horizontal over-crossings as follows: These moves produce an isotopic diagram to the original

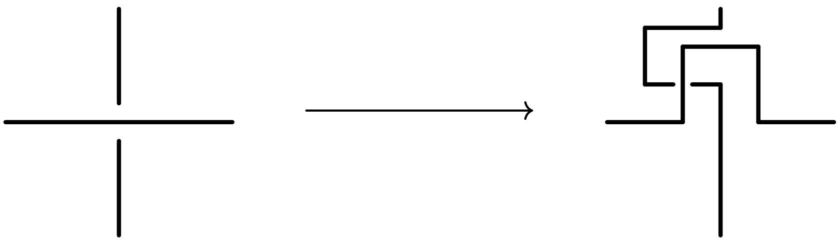

Figure 2: Modifying the horizontal crossing to make it a vertical crossing.

link diagram. Now, perturb the diagram again such that different vertical segments are not collinear, and repeat this for horizontal segments. Now, mark bends in the segments by ● and ○ such that vertical segments point from ● to ○, and horizontal segments point from ○ to ●.

Once this process is finished, the result is a valid grid diagram for the original oriented link L. □

Note the piecewise linear embedding in fig. 3. While we have given connecting segments between the circles in the grid diagrams (and will continue to do so to illustrate operations on the underlying knot), they are not necessary. Also note the inherent *oriented* nature of grid diagrams: every segment is oriented from a ○ to a ● horizontally, and the reverse for vertical strands.

[2]We do not lose any information in this process.

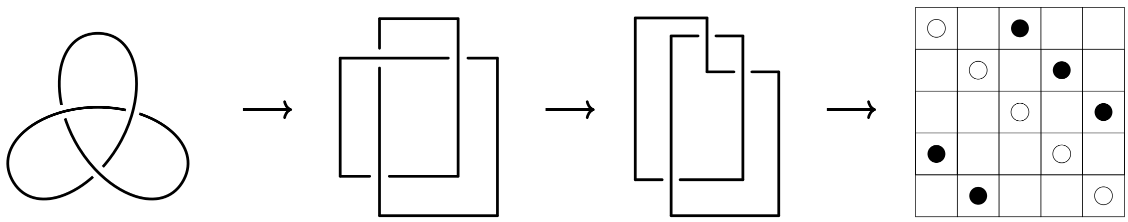

Figure 3: An example of the process described in the proof of theorem 1.3.

2 Grid Moves and Cromwell's Theorem

Much like with knot diagrams, we can define moves between two grid diagrams that preserve the underlying isotopy of the knots they represent. These are called **grid moves**.

We can define a notion of height of a row (respectively, column) by considering the intervals defined on the spaces $[0, n] \times \{0\}$ and $\{0\} \times [0, n]$, where n is the grid number of a grid diagram $\mathbf{G}$. Notice that for each row (and analogously, column), the $\circ$ and the $\bullet$ bound a subinterval of $[0, n]$.

Definition 2.1 (Commutation). Let $\mathbf{G}$ be a grid diagram. Consider two consecutive rows (respectively, columns) of $\mathbf{G}$. If the height intervals of these rows are disjoint, one is strictly contained in the other, or the endpoint of one interval is the beginning of the other, then interchanging these rows is called a **row commutation** (respectively, **column commutation**) move. Refer to fig. 4 for an example.

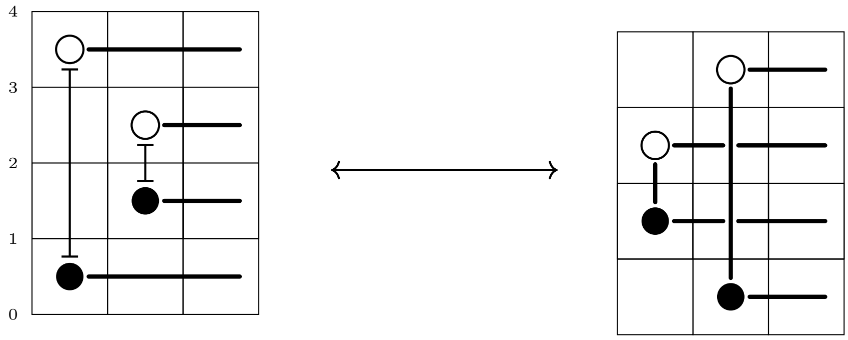

Figure 4: A column commutation. Note that this is a RII move on the underlying knot.

Much like Markov moves in braid theory, we have the notion of "introducing a kink" into a grid diagram as well.

Definition 2.2 (Stabilization). Let $\mathbf{G}$ be a grid diagram with grid number n. A grid diagram $\mathbf{G}'$ is a **stabilization** of $\mathbf{G}$ if it can be obtained by splitting a row and a column into two and has grid number $n + 1$, as follows: choose some marked square in $\mathbf{G}$ and erase the marking, along with its corresponding marked square in that row and the corresponding marked square in its column. Now split the row and column into two by adding a new horizontal and vertical line. After inserting new valid markings, the resulting diagram $\mathbf{G}'$ is a stabilization of $\mathbf{G}$.

Refer to fig. 5 for examples of stabilization. The inverse of a stabilization is called a **destabilization**.

Definition 2.3 (Grid Moves). The row and column commutations, stabilization, and destabilization are termed as **grid moves**.

We can now state Cromwell's Theorem.

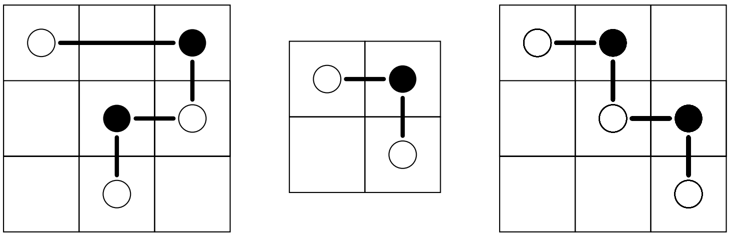

Figure 5: Stabilization.

Theorem 2.4 (Cromwell [Cro95]). Two grid diagrams represent isotopic links if and only if there is a finite sequence of grid moves that transforms one into the other.

One way to prove this is to show that grid moves are sufficient to generate the Reidemeister moves on their underlying knot. While there is some more subtlety to the proof, we can indeed generate Reidemeister moves from grid moves.

Notice that row and column commutation are the direct analogue of RII moves for grid diagrams, and hence generate RII moves on the underlying knot.

An RI move can be generated by a composition of a stabilization and a commutation as follows:

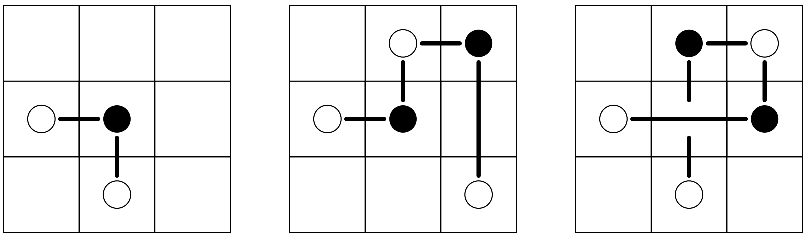

Figure 6: Generating an RI move by composing a stabilization and a commutation.

We can also generate RIII moves using commutation.

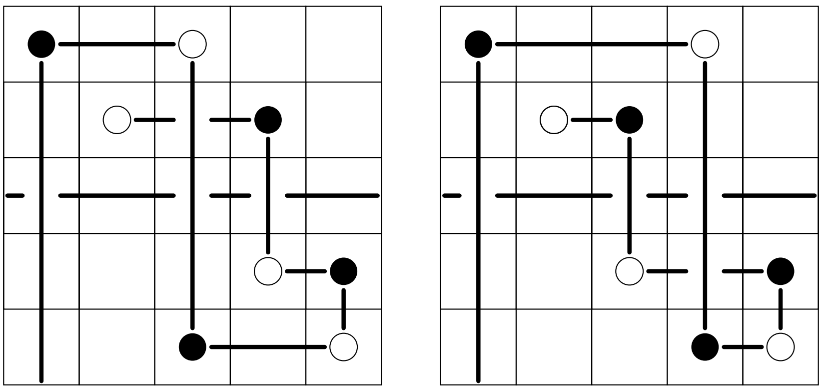

Figure 7: Generating an RIII move from a commutation.

3 Presentation of Knot Group

One application of grid diagrams is giving a simple presentation of the knot group $\pi_1(S^3 \setminus K)$.

Definition 3.1 (Knot Group). The fundamental group $\pi_1(S^3 \setminus K)$ of the complement of a knot $K \hookrightarrow S^3$ is known as the **knot group**. It is a link invariant.

To determine a presentation of the knot group from a grid diagram $\mathbf{G}$ with grid number n, we label the vertical columns in order from left to right as $c_1, c_2, \ldots, c_n$. Then, we label the vertical lines separating the rows of $\mathbf{G}$ as $r_1, r_2, \ldots, r_{n-1}$ from top to bottom. These correspond to relations r_i given by the order in which the r_i meet the c_i from left to right.

Theorem 3.2 ([OSS15]). The presentation

$$\langle c_1, c_2, \ldots, c_n \mid r_1, r_2, \ldots, r_{n-1} \rangle$$

is a presentation of the group $\pi_1(S^3 \setminus K)$.

To illustrate this, consider the (alternative to fig. 1) diagram of the right-handed trefoil $T(2,3)$ in fig. 8. Checking the intersections, we see that we have a presentation

$$\langle c_1, c_2, c_3, c_4, c_5 \mid c_1c_3, c_1c_2c_3c_4, c_1c_2c_4c_5, c_2c_5 \rangle.$$

This can be simplified. Noting that $c_3 = c_1^{-1}$, $c_2 = c_5^{-1}$, we can see that $c_4 = c_1c_2^{-1}c_1^{-1}$. Then we can simplify the presentation to

$$\langle c_1, c_2 \mid c_1c_2c_1 = c_2c_1c_2 \rangle \xrightarrow[c_2c_1c_2 \mapsto b]{c_1c_2 \mapsto a} \langle a, b \mid a^3 = b^2 \rangle.$$

Notice that the expression on the left is the Wirtinger presentation of $\pi_1(S^3 \setminus T(2,3))$.

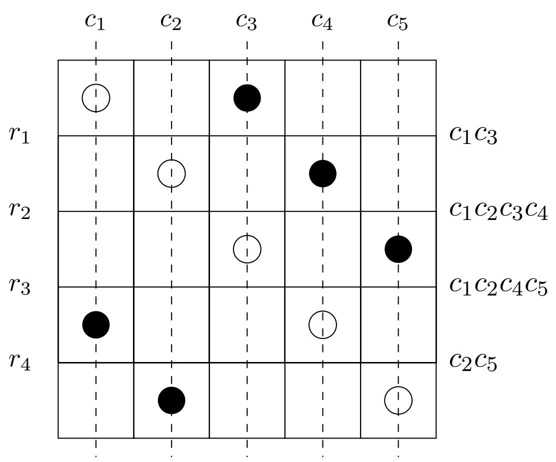

Figure 8: Computing the presentation of $\pi_1(S^3 \setminus T(2,3))$.

4 Monotonic Simplification

One interesting result is the following theorem of Dynnikov:

Theorem 4.1 (Dynnikov [Dyn06]). All grid diagrams of the unknot can be simplified monotonically.

This result is not true for knot diagrams in general[3]. To prove this, Dynnikov uses arc-presentations, which are a related concept to grid diagrams.

In particular, this means that the unknotting problem can be solved in polynomial time when done on grid diagrams.

[3] In fact, it is often the case that more crossings must be introduced into a diagram before it can be further simplified.

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