Posets

A First Look

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A Quick Introduction...

For those of you I haven't met before...

Here's a few things about me:

- Hi! I'm Pranav!
- I'll be a freshman in college this fall.
- My favorite math subject is combo!
- Check me out at `https://pranavkonda.com`!

But that's enough about me, what you're really here for is the math.

Some background about posets and notation

- Today we'll be learning about *partially ordered sets*.
- We also call them posets for short.
- They have cool connections to other areas of combo, like PIE.
- Since there's going to be some slightly advanced theory, let me know if I need to slow down or explain something in more detail.

The set $\{1, 2, \ldots, n\}$ is going to be denoted by $[n]$.

Basic Definitions

Binary Relations

As you probably already know, the *Cartesian Product* of two sets A and B, $A \times B$, is the set

$$\{(a, b) : a \in A, b \in B\}.$$

Definition (Binary Relation)

A binary relation R over two sets A and B is a subset of $A \times B$. We think about (a, b) as aRb, where we're *relating* a to b.

Posets

Now we're ready to define a poset! A poset is a set P, along with a binary relation $\leq$ that satisfies the following properties:

- **Reflexivity**: For any $s \in P$, $s \leq s$.
- **Antisymmetry**: If $t \leq s$ and $s \leq t$, then $s = t$.
- **Transitivity**: If $s \leq u$ and $u \leq t$, then $s \leq t$.

Just like regular less-than, $s < t$ indicates that $s \neq t$. Same for greater than signs.

Posets

The "crux" of what makes the poset a *partially ordered* set is that for any two elements in P, it's not necessary for $s \leq t$ or $t \leq s$ to hold! We have formal words for this (let $s, t \in P$):

- If $s \leq t$ or $t \leq s$, then we say s and t are **comparable**.
- Otherwise, we say s and t are **incomparable**.

Sometimes, our set and partial order will only have comparable elements, in which case we call it a total order (like $(\mathbb{R}, \leq)$ for example).

The n chain

- Suppose n is a positive integer. Then the set $[n]$ with the regular less-than relation forms a poset, **n**, which we call the *n*-**chain**. It's pretty easy to verify that this satisfies reflexivity, antisymmetry, and transitivity.
- We'll see why it's called a *chain* shortly, but we need to define a few things first.

Power sets

Your turn to prove something!

Problem

Let S be a set. Prove that $(\mathcal{P}(S), \subseteq)$ is a poset.

Isomorphism

As expected with any mathematical object, we have some notion of poset *isomorphism*, or an *order-preserving bijection*. We say two posets P and Q are isomorphic if there's some map $\phi : P \to Q$ such that

$$s \leq t \iff \phi(s) \leq \phi(t).$$

Pretty generic definition of isomorphism.

Subposets

Subposets are a bit wonky, since there are two ways we can define one. Since we will almost always use only one kind, we're going to go through that.

Definition (Induced subposet)

Let P be a poset. Then Q is an induced subposet of P if for $s, t \in Q$, $s \leq_Q t$ if and only if $s \leq_P t$. Obviously, $Q \subseteq P$.

Covering relations

This is an important definition. For two elements s and t in P, we say that t *covers* s if $s < t$ and there does not exist a $u \in P$ such that $s < u < t$. We denote these *covering relations* by $s \lessdot t$.

Covering relations are important because they let us draw posets! Formally, we call these drawings *Hasse diagrams*, and they're basically directed acyclic graphs but we understand every edge points "up".

The n-chain returns

Let's consider the n-chain again, by drawing its Hasse diagram for $n = 4$. Obviously, $n \lessdot n + 1$, so we get something like this:

A lot of posets look really cool when we draw them.

Power sets again

Your turn again! Try drawing the Hasse diagram of $(\{1, 2, 3\}, \subseteq)$. We also have a formal term for the poset $([n], \subseteq)$: the *Boolean poset*, B_n. Be sure to get familiar with this poset, as it's going to pop up again and again.

Chains and Generating Functions

Chains, in general

- Chains are actually a more general component of posets.
- A chain is a poset that is totally ordered: every element is comparable. So $\mathbb{R}$ and $\mathbb{Z}$ are chains.
- A subset $C \subset P$ is a chain if every element in it is comparable, this is the more useful definition.
- We also have a notion of *length*: the length of a chain $C \subset P$ is $\ell(C) = |C| - 1$.

Rank and graded posets

The *rank* of a poset P is the length of its maximal chain:

$$\operatorname{rank} P = \max_{C \subset P} \ell(C).$$

When every maximal chain (meaning there isn't a larger chain that contains it) of P has the *same* length, we call P a *graded* poset. If that length is n, then P is graded of rank n.

Rank functions

If P is a graded poset of rank n, then there exists a *rank function* $\rho : P \to \{1, 2, \ldots, n\}$, which satisfies these special properties:

- If s is a minimal element of P, then $\rho(s) = 0$.
- If $s \lessdot t$, then

$$\rho(s) + 1 = \rho(t).$$

If $\rho(s) = i$, then we say s has rank i.

Rank generating functions

Suppose that P is a graded poset of rank n. Then there exists a *rank generating function*

$$F(P, x) = \sum_{k=0}^{n} p_k x^k,$$

where p_k is the number of elements in P with rank k.

Rank generating functions

Can you try finding the rank function and rank generating function for the following posets?

- The n-chain $\mathbf{n}$.
- The boolean poset B_n.
- This is the other major poset I wanted to talk about: the *Divisor poset* D_n. For this poset, we take a positive integer n, and let the ground set be the positive integer divisors of n. Then $s \leq t$ if $s|t$.

Antichains

- An *antichain* is a subset $A \subset P$ such that *no* two elements of A are comparable.
- Antichains are also called **Sperner families**, as they have a very deep connection to **Sperner's theorem** in extremal combinatorics.
- We'll look at some results related to extremal combinatorics (Dilworth's and Mirsky's theorems) some other time.

Order Ideals

Definition (Order Ideals)

An **order ideal** (also called a down set) is a subset $I \subset P$ such that if $t \in I$ and $s \leq t$, $s \in I$.

- There's a *dual* of an order-ideal, called an *up-set*, where we replace the $s \leq t$ condition with $s \geq t$.
- For experts, this should remind you of the definition of a regular ideal.

A nice problem

Problem

Show that the number of order ideals in P is equal to the number of antichains in P.

Hint: Argue for a bijection!

References and places to learn more

- *R. Stanley, Enumerative Combinatorics Volume 1.*
- One of the best combo textbooks out there, although it's very dense and reference-like.
- More of these lectures!